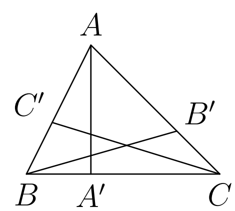
# Chapter 1

# Theorems of Ceva and Menelaus

We start these lectures by proving some of the most basic theorems in the geometry of a planar triangle. Let $A, B, C$ be the vertices of the triangle and $A', B', C'$ be any points on the lines $BC, CA$ and $AB$ respectively.

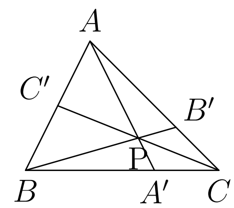



For arbitrarily chosen points $A', B', C'$ there is no reason to expect that the three segments $AA', BB', CC'$ will meet at one point. However they do meet in many cases of geometrical significance, e.g. the three angle bisectors meet at one point. Before we discuss these special cases, we will formulate a general criterion that tells us when the segments $AA', BB', CC'$ are concurrent. This criterion is called "Ceva's theorem".

**Theorem 1** (Ceva)**.** *Let $ABC$ be a triangle and $A', B', C'$ be points on the sides $BC, CA$ and $AB$ respectively. Three lines $AA', BB', CC'$ are concurrent if and only if*

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$$

We will prove this theorem using ratios of the areas of the triangles created by the segments $AA', BB'$ and $CC'$.




We will rely on the following lemma:

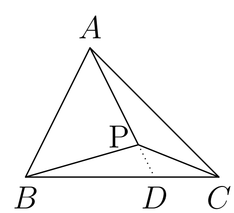
**Lemma 2.** *Let $ABC$ be an arbitrary triangle and let $D$ be a point on the side $BC$. Then* $\frac{Area(BDA)}{Area(DCA)} = \frac{BD}{DC}$

*Proof.* Triangles $BDA$ and $DCA$ share a common height from the vertex $A$. Let $h$ be the length of this height. Then $\text{Area}(BDA) = \frac{1}{2}h \cdot BD$, $\text{Area}(DCA) = \frac{1}{2}h \cdot DC$ and hence $\frac{\text{Area}(BDA)}{\text{Area}(DCA)} = \frac{BD}{DC}$. □

We will in fact use the following corollary from this lemma.

**Corollary 3.** *Let $ABC$ be a triangle with a point $D$ lying on the side $BC$ and point $P$ — any point on the segment $AD$. Then*

$$\frac{Area(BPA)}{Area(PCA)} = \frac{BD}{DC}$$




*Proof.* From lemma 2 applied to triangle $ABC$ we get

$$\text{Area}(BDA) = \frac{BD}{DC} \cdot \text{Area}(DCA)$$

From the same lemma applied to triangle $BCP$ we get

$$\text{Area}(BDP) = \frac{BD}{DC} \cdot \text{Area}(DCP)$$

By subtracting these two from each other we get

$$\text{Area}(BPA) = \frac{BD}{DC} \cdot \text{Area}(PCA)$$

as needed. $\square$

We are ready now to prove Ceva's theorem.

*Proof.* For one of the implications suppose that the segments $AA', BB', CC'$ meet at a point $P$. From corollary 3 we find that

$$\frac{BA'}{A'C} = \frac{\text{Area}(BPA)}{\text{Area}(PCA)}$$
$$\frac{CB'}{B'A} = \frac{\text{Area}(CPB)}{\text{Area}(PAB)}$$
$$\frac{AC'}{C'B} = \frac{\text{Area}(APC)}{\text{Area}(PBC)}$$

Multiplying these equations we get that $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$, as needed.

For the other implication, assume that points $A', B', C'$ on the sides $BC, CA, AB$ satisfy $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$. Let $P$ be the point of intersection of $AA'$ and $BB'$. Extend the segment $CP$ to meet $AB$ at $C''$. Then, by the direction we have already proved,

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC''}{C''B} = 1$$

But we assumed also that

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$$

Hence $\frac{AC'}{C'B} = \frac{AC''}{C''B}$, meaning that $C'$ and $C''$ are in fact the same point. Thus the point $P$ lies on the segment $CC'$ as well. $\square$

**Remark 1.** *Ceva's theorem has analogues both in spherical and hyperbolic geometries. Instead of quotients of lengths of sides, quotients of certain functions of these lengths appear there (sines for spherical geometry and hyperbolic sines for hyperbolic geometry). The proof we supplied can be adapted to these geometries as well, except other formulas for areas of triangles will be needed.*

We can also find the ratio in which the point $P$ divides the cevians $AA', BB'$ and $CC'$ in terms of the ratios in which the points $A', B'$ and $C'$ divide the sides of the triangle. Indeed, it follows from the lemma that $\frac{AP}{PA'} \cdot \text{Area}(A'PC) = \text{Area}(PCA)$ and similarly that $\frac{AP}{PA'} \cdot \text{Area}(BA'P) = \text{Area}(BPA)$. By adding these two equalities we get that $\frac{AP}{PA'} \cdot \text{Area}(BCP) = \text{Area}(PCA) + \text{Area}(BPA)$, or $\frac{AP}{PA'} = \frac{\text{Area}(PCA)}{\text{Area}(BCP)} + \frac{\text{Area}(BPA)}{\text{Area}(BCP)}$. But from the corollary we know that $\frac{\text{Area}(PCA)}{\text{Area}(BCP)} = \frac{AC'}{C'B}$ and $\frac{\text{Area}(BPA)}{\text{Area}(BCP)} = \frac{AB'}{B'C}$. Thus

$$\frac{AP}{PA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C}$$

We will now apply Ceva's theorem to prove a couple of classical results. Recall that a median in a triangle is a line connecting a vertex with the midpoint of the opposite side. An angle bisector is a line dividing an internal angle of a triangle at a vertex into two equal parts. An altitude is a perpendicular dropped from a vertex to the opposite side of the triangle.

We will prove that the three medians of any triangle are concurrent. The same holds for angle bisectors and altitudes. We will first derive these results from Ceva's theorem and then rederive them in an independent way.

**Medians** If $A', B', C'$ are midpoints of the sides of the triangle $ABC$, then $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1 \cdot 1 \cdot 1 = 1$, and hence, by Ceva's theorem the three medians $AA', BB'$ and $CC'$ are concurrent.

Let's prove the same result more directly.

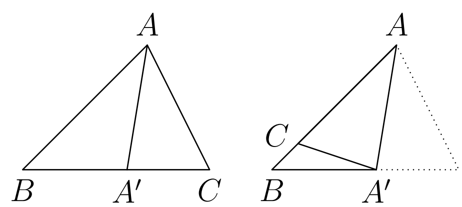
We claim that the median $AA'$ is exactly the locus of points $P$ in the plane such that $\text{Area}(ABP) = \text{Area}(PCA)$. Indeed, if the point $D$ is the intersection point of $AP$ and $BC$, then the quotient $\frac{\text{Area}(ABP)}{\text{Area}(PCA)}$ is equal to $\frac{BD}{DC}$ by corollary 3. Hence it can be equal to 1 if and only if the point $D$ coincides with the midpoint of $BC$; that is $P$ lies on $AA'$.

Now let $P$ be the point of intersection of medians $AA'$ and $BB'$. By the result we just showed $\text{Area}(BPA) = \text{Area}(PCA)$ (because $P$ is on $AA'$) and $\text{Area}(BPA) = \text{Area}(BCP)$ (because $P$ is on $BB'$). Combining these two equalities we get that $\text{Area}(PCA) = \text{Area}(BCP)$, hence the point $P$ lies on the median $CC'$ as well.

**Angle bisectors** To prove that three angle bisectors meet at one point, we will use the following lemma:

**Lemma 4.** *Let $A'$ be a point on the side $BC$ so that $AA'$ is the internal angle bisector of the angle at vertex $A$ in triangle $ABC$. Then $\frac{BA'}{A'C} = \frac{BA}{AC}$.*

*Proof.* From lemma 2 we know that $\frac{BA'}{A'C} = \frac{\text{Area}(BA'A)}{\text{Area}(A'CA)}$. Now the equality of angles $\angle BAA'$ and $\angle A'AC$ implies that if we reflect the triangle $A'CA$ in the line $AA'$, then the reflected copy will share an angle with triangle $BA'A$ and the reflection of $C$ will lie on the line $AB$ (see pic.).

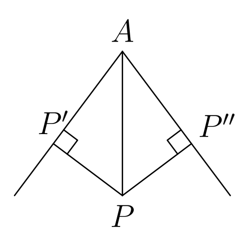



But in the picture after the reflection we see that $\frac{\text{Area}(BA'A)}{\text{Area}(A'CA)} = \frac{AB}{AC}$. □

This lemma implies that if $AA', BB'$ and $CC'$ are internal bisectors of triangle $ABC$, then $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{BA}{AC} \cdot \frac{CB}{BA} \cdot \frac{AC}{CB} = 1$, so Ceva's theorem tells us that $AA', BB'$ and $CC'$ are concurrent.

We can prove it in a different way as well. For this we first notice that the internal angle bisector of an angle with vertex $A$ is exactly the locus of points $P$ inside the angle that are equidistant from the two rays that form the angle. Indeed, let $P'$ and $P''$ be the feet of perpendiculars dropped from the point $P$ to the two rays. If the point $P$ is on the angle bisector, then the reflection of triangle $APP'$ in line $AP$ must coincide with the triangle $APP''$ (because $\angle PAP' = \angle PAP''$ and $\angle APP' = \pi/2 - \angle PAP' = \pi/2 - \angle PAP'' = \angle APP''$), and hence $PP' = PP''$.

Conversely, if $P$ is a point inside the angle $A$ with $PP' = PP''$, then in the right triangles $APP'$ and $APP''$ side $AP$ is common, and sides $PP'$ and $PP''$ are equal. But since there is only one right triangle with a given leg and hypothenuse, the triangles must be equal. In particular this implies that $\angle PAP' = \angle PAP''$, so the point $P$ lies on the angle bisector of angle $A$.

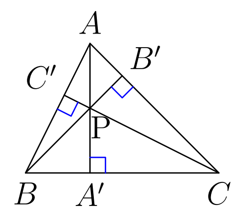



Now let $AA'$, $BB'$ and $CC'$ be the internal angle bisectors of triangle $ABC$. Let $P$ be the point of intersection of the angle bisectors $AA'$ and $BB'$.

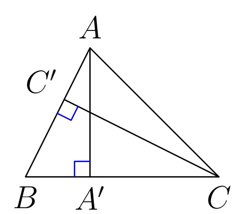
We have proved above that point $P$ is equidistant from sides $AB$ and $AC$ because it lies on angle bisector $AA'$. It is also equidistant from sides $AB$ and $BC$, because it lies on the angle bisector $BB'$. Hence the point $P$ is located at the same distance from the sides $AC$ and $BC$. It must then lie on the angle bisector $CC'$.

Note that since the point of intersection of the angle bisectors of triangle $ABC$ is equidistant from the sides of the triangle, it must be the center of the circle inscribed in triangle $ABC$.

As another corollary from Ceva's theorem, we can prove that the three altitudes of any triangle are concurrent (for obtuse triangles the point of concurrency lies outside the triangle, so some care must be taken in this case).




Indeed, if $AA'$ and $CC'$ are altitudes in a triangle $ABC$, then the triangles $ABA'$ and $CBC'$ are similar, since they share an angle at $B$ and are both right-angled. Hence $\frac{BA'}{C'B} = \frac{BA}{CB}$.

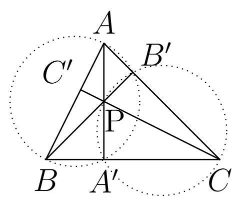



Applying this argument three times we get that the feet of the altitudes satisfy $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{BA'}{C'B} \cdot \frac{CB'}{A'C} \cdot \frac{AC'}{B'A} = \frac{BA}{CB} \cdot \frac{CB}{AC} \cdot \frac{AC}{BA} = 1$. Ceva's theorem then implies that the three altitudes are concurrent.

Now we will reprove this result without reference to Ceva's theorem. Instead we will apply a technique called angle chasing — repeatedly noting relations between different angles until we get what we need.

Let $AA'$ and $BB'$ be altitudes of triangle $ABC$ and let point $P$ be their point of intersection. Connect points $C$ and $P$ by a line and let $C'$ be the point of intersection of this line with side $AB$. We want to show that angle

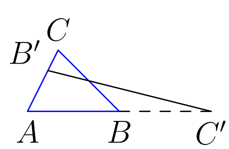
$\angle CC'B$ is right. For this note that $\angle AB'B = \angle AA'B = \pi/2$, so both points $B'$ and $A'$ lie on the circle having $AB$ as a diameter. The angles $\angle BAA'$ and $\angle BB'A'$ are supported by the same arc in this circle, and therefore they are equal. Similarly, since both angles $\angle PA'C$ and $\angle PB'C$ are right, the points $A'$ and $B'$ lie on the circle having $PC$ as a diameter. Angles $\angle A'CP$ and $\angle A'B'P$ are supported by the same arc in this circle, and hence are equal. Combining these two results we get that $\angle BAA' = \angle BB'A' = \angle PB'A' = \angle PCA' = \angle C'CB$. This means that in triangles $BAA'$ and $BCC'$ the angle at $B$ is common, and $\angle BAA' = \angle BCC'$, so the other two angles must be equal as well: $\angle BC'C = \angle BA'A = \pi/2$, as required.




*Add exercise proving the theorem from the corresponding theorem about side bisectors.

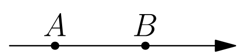
There is a theorem, which is very close to Ceva's theorem in spirit and in formulation, called Menelaus' theorem. Let's see what it states:

**Theorem 5.** *Let $ABC$ be a triangle and let points $A', B'$ and $C'$ belong to the lines $BC, CA$ and $AB$ respectively. The points $A', B'$ and $C'$ belong to one line if and only if* $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$




The reader may object that there must be some mistake in this formulation, because the condition $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$ from Menelaus' theorem is exactly the same as the condition $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$ that appeared in Ceva's theorem. If both theorems are correct, which one applies to any given triangle? One way to distinguish the cases in which the two theorems apply is to count the number of points $A', B'$ and $C'$ that lie outside of the corresponding sides $BC, CA$ and $AB$ - if none or two, then Ceva's theorem applies; if one or three - then Menelaus' applies. It is, however, much more pleasing (both aesthetically and for generalizing both theorems) to treat both formulations on common grounds. For this we introduce the notion of oriented lengths.

To speak of oriented length we need a directed line (that is a line with a choice of one of the two possible directions of movement along it) and a directed segment on it (that is a segment for which we can distinguish its beginning and its end). The oriented length of a segment from point $A$, on a directed line, to point $B$ on the same line, is just its usual length, if the direction from $A$ to $B$ agrees with the chosen direction along the line; otherwise, it is the usual length taken with sign minus.




The oriented length of the segment from $A$ to $B$ on the oriented line from figure 1 is positive, while the oriented length of the segment from $B$ to $A$ is negative. Notice that if we change the orientation of the line to the opposite one, the signs of the lengths will change as well.

For a segment on a non-oriented line the sign of the length doesn't have any meaning (there is no reason to prefer moving in one of the directions along a line to moving in the other). However when we have two segments on the same line, the quotient of the oriented lengths makes sense: we can choose either direction along the line and compute the quotient of oriented lengths. If we choose the other direction instead, both oriented lengths will change sign, but their quotient won't change.

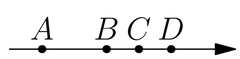



Figure 1: While the oriented lengths of segments $AB$ and $CD$ don't have any meaning without the choice of direction along the line containing them, their quotient $\frac{AB}{CD}$ is well defined. In fact $\frac{AB}{CD} = \frac{BA}{DC} = -\frac{AB}{DC} = -\frac{BA}{CD}$

Now, in the formulations of Ceva's and Menelaus' theorems the conditions $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$ and $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$ should be regarded as conditions on quotients of oriented lengths. When considered this way, they are indeed different: for Ceva's theorem the condition is $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$ while for Menelaus' theorem it is $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$;

**Question 1.** *Why does the condition* $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$ *of Ceva's theorem imply that either none or two of the points* $A', B'$ *and* $C'$ *lie outside the* *corresponding segments $BC, CA$ and $AB$? How many of them should lie outside the segments $BC, CA$ and $AB$ if the condition of Menelaus' theorem applies?*

Having clarified the formulations of Ceva's and Menelaus' theorems, we proceed to proving the latter. The following lemma will be helpful to us:

**Lemma 6.** *Let $l_1, l_2$ be two lines. Let points $A_1, B_1, C_1, D_1$ belong to line $l_1$ and points $A_2, B_2, C_2, D_2$ - to line $l_2$. Suppose that the lines $A_1A_2, B_1B_2, C_1C_2$ and $D_1D_2$ are parallel. Then $\frac{A_1B_1}{C_1D_1} = \frac{A_2B_2}{C_2D_2}$.*

This lemma can be interpreted as saying that the quotient of oriented lengths of segments is preserved under parallel projection from one line to another.

*Proof.* Consider the case when lines $l_1, l_2$ are parallel first. If we translate the line $l_1$ so that the point $A_1$ gets translated to point $A_2$, the points $B_1$,$C_1$ and $D_1$ get translated to points $B_2, C_2$ and $D_2$. Of course, translation preserves not just the quotients of lengths, but the lengths themselves.

Suppose then that the lines $l_1$ and $l_2$ intersect at a point $O$. In that case considerations of similar triangles show that there is a number $k$ so that $k = \frac{OA_1}{OA_2} = \frac{OB_1}{OB_2} = \frac{OC_1}{OC_2} = \frac{OD_1}{OD_2}$. But then $\frac{A_1B_1}{C_1D_1} = \frac{OB_1 - OA_1}{OD_1 - OC_1} = \frac{k(OB_2 - OA_2)}{k(OD_2 - OC_2)} = \frac{A_2B_2}{C_2D_2}$ as we wanted to show.

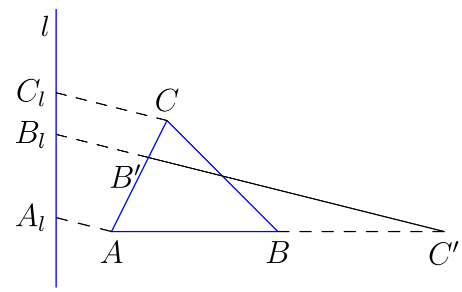
□

Now we can prove the theorem of Menelaus.

*Proof.* Suppose that the points $A', B', C'$ lie on one line. Choose any line $l$ which is not parallel to it, and consider a parallel projection of all the picture to the line $l$ along the line $A'B'C'$. Then the points $A', B', C'$ project to the same point $O$ on the line $l$. Denote by $A_l, B_l, C_l$ the images of points $A, B, C$ under this projection. Then the lemma implies that $\frac{A'B}{A'C} = \frac{OB_l}{OC_l}, \frac{B'C}{B'A} = \frac{OC_l}{OA_l}, \frac{C'A}{C'B} = \frac{OA_l}{OB_l}$. By multiplying these three equalities we get that $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \frac{OB_l}{OC_l} \cdot \frac{OC_l}{OA_l} \cdot \frac{OA_l}{OB_l} = 1$ (because of cancellation).

The proof of the converse direction is similar to the proof of the converse direction in Ceva's theorem: suppose $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$ and let $C''$ be the point on line $AB$ at which the line $A'B'$ intersects it. Then what we proved shows that $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C''A}{C''B} = 1$ and hence $\frac{C''A}{C''B} = \frac{C'A}{C'B}$, showing that $C'' = C'$; that is the point $C'$ lies on the line $A'B'$.

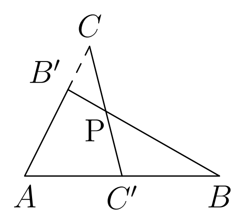
□



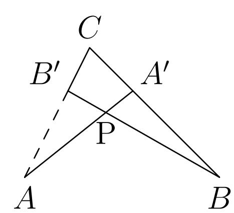

## 1.1 The relationship between Ceva's and Menelaus' theorems

One may wonder whether the results of Ceva and Menelaus, which are so similar in nature and formulation, can be deduced from each other. And indeed it is the case. Suppose we know Menelaus' theorem and want to deduce Ceva's: let $ABC$ be a triangle and let cevians $AA', BB'$ and $CC'$ meet at a point $P$.

Then, the application of Menelaus' theorem to triangle $AB'B$ and line $C'PC$ gives that $\frac{AC'}{C'B} \cdot \frac{BP}{PB'} \cdot \frac{B'C}{CA} = -1$.




The application of Menelaus' theorem to triangle $B'BC$ and line $APA'$ gives $\frac{BA'}{A'C} \cdot \frac{CA}{AB'} \cdot \frac{B'P}{PB} = -1$.

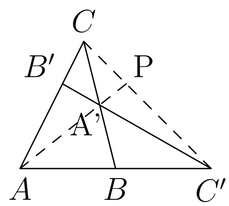



By multiplying these equations we get $\frac{AC'}{C'B} \cdot \frac{BP}{PB'} \cdot \frac{B'C}{CA} \cdot \frac{BA'}{A'C} \cdot \frac{CA}{AB'} \cdot \frac{B'P}{PB} = 1$. Notice the ratios $\frac{BP}{PB'}$ coming from the first product and $\frac{B'P}{PB}$ coming from the second product are inverses of each other. Notice also that the side $CA$

appears in the numerator in the first product and in the denominator in the second one. Thus cancelling like terms we find $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$. This proves Ceva's theorem.

For the derivation of Menelaus' theorem from the theorem of Ceva we refer the reader to exercise 1

**Exercise 1.** *In this exercise we are going to deduce Menelaus' theorem from corollary of Ceva's theorem.*




*Let $ABC$ be a triangle and let $A', B', C'$ be three collinear points on lines $BC, AC, AB$ respectively. Let $P$ be the point of intersection of lines $BB'$ and $AA'$. 1) In triangle $ABB'$ the Cevians $AP, BC, B'C'$ are concurrent. Verify that the following equalities follow from Ceva's theorem and its corollary:*

$$\frac{BP}{PB'} \cdot \frac{B'C}{CA} \cdot \frac{AC'}{C'B} = 1$$

$$\frac{BA'}{A'C} = \frac{BC'}{C'A} + \frac{BP}{PB'}$$

*2) Deduce from the two equations above that* $\frac{BA'}{A'C} = \frac{BC'}{C'A}(1 + \frac{CA}{B'C}) = \frac{BC'}{C'A} \cdot \frac{B'A}{B'C}$ *or* $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$.