Chapter 1

Projective duality

1.1 Definition of projective duality

We are going to see now a very funny transformation, called projective duality. What it allows us to do is to take a projective theorem (like Pascal's), apply this transformation, and get a new theorem, which is guaranteed to be true! We saw something similar with inversion: we could prove a theorem, apply inversion and get a new one. Projective duality, however, is not a construction that requires any choices (like the choice of circle for inversion), but rather is an internal feature of projective geometry.

Let us proceed straight to the definition. Let V be a vector space and let V^* denote the space dual to V, i.e. the space of all linear functions on V. First we define duality as a transformation from the set of linear subspaces of V to the set of linear subspaces of V^*: the space $W \subset V$ gets sent to the space $W^* = \{f \in V^* | \langle f, w \rangle = 0\}$, i.e. to the space of all functionals evaluating to zero on W. We used the notation $\langle f, w \rangle$ for the value of functional f on vector w to emphasize the symmetry of roles of f and w: we can think of a vector w as a functional on the space V^* evaluating on the element $f \in V^*$ to $\langle f, w \rangle$. This way of thinking shows how to identify $(V^*)^*$ with V itself (for finite dimensional V the embedding of V into $(V^*)^*$ we described is an isomorphism, because of equality of dimensions of the space and its dual). If we identify V with $(V^*)^*$ in this way, applying duality twice becomes identity: $(W^*)^* = W$ for any $W \subset V$.

The duality we described is easily seen to reverse inclusions: if the linear subspace W_1 contains W_2, then every function vanishing on it, must vanish on W_2 as well, hence W_1^* is contained in W_2^*.

Now we can projectivise duality we defined above to get the notion of projective duality: projective duality is the transformation from projective subspaces of $\mathbb{P}V$ to projective subspaces of $\mathbb{P}V^*$ sending the subspace $\mathbb{P}W$ (for $W \subset V$ a linear subspace) to $\mathbb{P}W^*$.

Since in the usual duality lines through origin get sent to hyperplanes through origin, in projective duality points are dual to hyperplanes. More generally spaces of dimension k are dual to spaces of codimension $k+1$.

By now we learned the following important properties of projective duality: it sends points to hyperplanes, hyperplanes to points and reverses inclusion relations.

For instance if we had a picture of n points lying on the same hyperplane before the duality, after applying duality, we will get the picture of n hyperplanes passing through the same point.

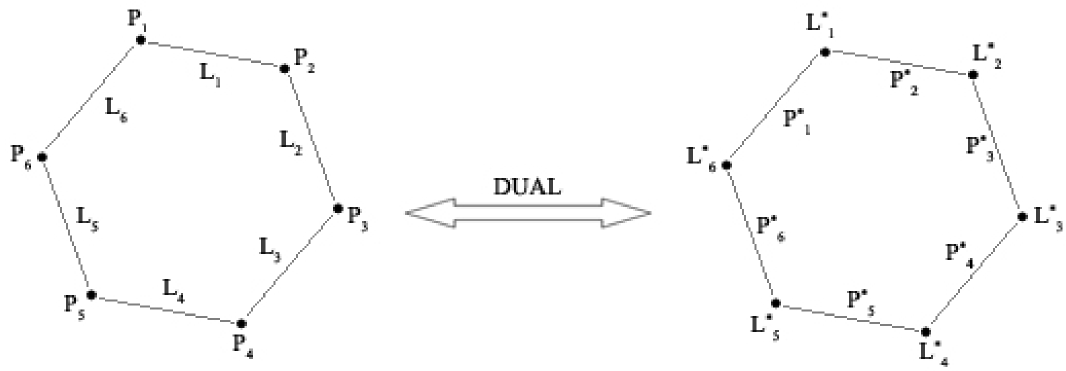
To get some really powerful applications of projective duality we will have to extend its definition to more than just linear subspaces.

1.2 Dual polygon and dual curve

We now restrict our attention to the projective plane.

Let C be a polygon in $mathbbRP^2$ whose vertices are $P_1, \dots, P_n$. Let $L_1, \dots, L_n$ denote the lines on which the sides of polygon C lie, so that L_i is the line through P_i and P_{i+1} and P_i is the point of intersection of lines L_{i-1} and L_i.

We define the dual polygon C^* to be the polygon whose vertices are points L_i^* - the points in $\mathbb{RP}_2^*$, which are dual to the sides of the polygon C. What are its sides? The side containing the points L_{i-1}^* and L_i^* should be dual to the point of intersection of L_{i-1} and L_i, i.e. to P_i.



Thus we can think of the dual polygon in two ways: as the polygon whose sides are the lines dual to the vertices of the original polygon, or as the polygon, whose vertices are the points dual to the sides of the original one.

By combining these two descriptions it's easy to see that the dual polygon of the dual polygon is the original one.

Indeed, the vertices of the dual polygon of C^* are the points dual to P_i^*, hence are equal to P_i.

We introduced the notion of a dual polygon to help us understand better the following definition of a dual curve.

Suppose C is a curve in the projective plane $\mathbb{RP}^2$. We define the dual curve to C to be the curve C^*, whose points are the points dual to tangent lines to C.

We can think of the space $\mathbb{RP}^2$ as the space parameterizing lines in $\mathbb{RP}^2$.Then the curve C^* should be thought of as the curve whose points are tangent lines to C.

We can imagine the curve C as a limit of polygons, whose sides get shorter and shorter and whose vertices lie on the curve C. Then the sides of this polygon tend to the lines tangent to the curve C. We see then that the dual curve C^* is approximated well by the dual polygons to the polygons approximating C.

The observations we made about dual polygons have their direct analogues for dual curves. Namely, dual curve to a curve C can be thought of in two ways:

The curve whose points are dual to the tangent lines of C

or

The curve whose tangent lines are dual to the points of C

This observation lead immediately to the claim that the dual curve of the dual curve of C is C itself.

Let's see a couple of examples: if the curve C has a double tangent, i.e. a line tangent to C at two points, then the dual curve will have a self-intersection point. Indeed, imagine a point P going along C. Then the tangent line at P is changing with P. As P goes from one tangency point of the double tangency line to C to the other one, the tangent line makes a loop and comes back to itself. Thus the dual curve intersects itself before closing up.

Dually, if a curve has a double point, then its dual has a double tangent.

Space for animation! (worth 5 points if done with asymptote animation module: the animation should depict a curve with point P moving along it and another curve C^*, which is dual to the first, on which point dual to the tangent line at P is moving)

In a similar vein, if a curve has an inflection point (i.e. point where its concavity changes), then the dual curve will have a cusp. (a cusp is a singularity point on a curve at which tangent line is still defined - if you imagine a car going along this curve, at a cusp point it starts going in reverse). As point P moves along C and passes the inflection point, the tangent line changes the direction of its movement.

Dually, if a curve has a cusp then the dual curve has an inflection point.

These two last statements make it necessary to define a little bit more precisely what kind of curves we are talking about. Talking about smooth curves is not enough - even if the original curve is smooth, its dual might have cusps. On the other hand we do need the notion of the tangent line at every point to define the dual of the curve at all. (*The class of curves, whose only singularities are cusps and double points seems to be the most natural class to talk about.*) In particular all algebraic curves fall into this class, and hence projective duality becomes a valuable tool in the study of projective algebraic geometry.

We could have defined the notion of dual hypersurface to a given hypersurface C in $\mathbb{RP}^n$ in essentially the same way we did it for curves. The points of the dual hypersurface C^* are the dual points of tangent hyperplanes to C. Then we could prove that the tangent hyperplanes to C^* are the hyperplanes dual to the points of C. We won't use this theorem in what follows, however, so we don't explain why it is true.

1.3 Dual to a quadric

In this section we want to understand what the dual hypersurface to a quadric is.

Let A be a symmetric matrix defining the quadric $\{x|\langle Ax, x\rangle = 0\}$. What is the equation of the hyperplane tangent to this quadric at point x_0 on it? To find out, we can move the point x_0 infinitesimally in direction ξ and see whether we go off the quadric or stay on it. To do so we evaluate the function $\langle Ax, x\rangle$ at point $x_0 + \xi$:

$$\langle A(x_0 + \xi), x_0 + \xi\rangle = \langle Ax_0, x_0\rangle + \langle A\xi, x_0\rangle + \langle Ax_0, \xi\rangle + \langle A\xi, \xi\rangle$$

The first term $\langle Ax_0, x_0\rangle$ in this formula vanishes because the point x_0 lies on the quadric. The last term is an infinitesimal of second order (it is of order $|\xi|^2$) and hence can be ignored in the linear approximation. Thus if we want to stay on the quadric to the first order of magnitude, we should have $\langle A\xi, x_0\rangle + \langle Ax_0, \xi\rangle = 0$. Since the matrix A is symmetric, $\langle A\xi, x_0\rangle = \langle \xi, Ax_0\rangle$ and thus the equation of the hyperplane tangent to the quadric at point x_0 is $\langle Ax_0, \xi\rangle = 0$. The point dual to this hyperplane clearly is Ax_0.

As the point x varies along the the quadric $\langle Ax, x\rangle = 0$, the point $y = Ax$ varies along the quadric $\langle y, A^{-1}y\rangle = 0$. Thus we have proved the following theorem.

Theorem 1. *The dual hypersurface to a non-degenerate quadric is a non-degenerate quadric. Moreover, the dual of the quadric given by equation* $x \in \mathbb{RP}^n|\langle Ax, x\rangle = 0$ *is the quadric given by equation* $y \in \mathbb{RP}^{n*}|\langle A^{-1}y, y\rangle = 0$.

1.4 Brianchon's theorem

Projective duality gives us a very simple way to generate new theorems from old ones: all we have to do is apply duality.

Let's do it for Pascal's theorem.

Recall that Pascal's theorem tells us that if $O_1, O_2, O_3, P_1, P_2, P_3$ are points on a quadric $\mathfrak{E}$, then the points E_1, E_2, E_3 defined as the points of intersection of O_2P_3 with O_3P_2, of O_1P_3 with O_3P_1 and of O_2P_1 with O_1, P_2 respectively, are collinear.

Let's try to apply duality to this claim.

The dual to a quadric $\mathfrak{C}$ is a quadric $\mathfrak{C}^*$. The dual lines to the points $O_1, O_2, O_3, P_1, P_2, P_3$ are lines $O_1^*, O_2^*, O_3^*, P_1^*, P_2^*, P_3^*$, which are tangent to the quadric $\mathfrak{C}^*$.

The point dual to the line O_2P_3 is the point of intersection of lines O_2^* and P_3^*. Similarly the point dual to the line O_3P_2 is the point of intersection of lines O_3^* and P_2^*. Thus the line dual to point E_1 is the line E_1^* containing the points of intersection of O_2^* with P_3^* and of O_3^* with P_2^*. The definition of E_2^* and E_3^* is similar.

Finally the statement dual to "E_1, E_2, E_3 are collinear" is "E_1^*, E_2^*, E_3^* are concurrent.

What we get is the wonderful theorem of Brianchon.

Theorem 2. *The diagonals connecting opposite vertices of a hexagon circumscribed around a conic are concurrent.*

Notice that we know this theorem is valid because we know the theorem of Pascal and projective duality. We could, however, prove this theorem without using projective duality, if we wanted to. To do so, we would need to dualize all the arguments we used for proving Pascal's theorem.

The key argument for the proof of Pascal's theorem was the following:

Let $f : l_1 \to l_2$ be a mapping from line l_1 to line l_2 in a projective plane. Suppose f is projective, (i.e. preserves cross ratios). Let O be the point of intersection of l_1 and l_2. Suppose that $f(O) = O$. Then there exists a point C in the plane so that the mapping f coincides with the central projection from line l_1 to line l_2 with center C.

In the dual statement, instead of a mapping taking in points on line l_1 and spitting out points of line l_2, we should talk about a mapping that takes in lines passing through one point and spitting out lines passing through another point.

So our dual statement will start with "Let $f : \mathfrak{P}_1 \to \mathfrak{P}_2$ be a mapping from the pencil $\mathfrak{P}_1$ of lines passing through point P_1 to the pencil $\mathfrak{P}_2$ of lines passing through point P_2. Suppose that f is projective (i.e. preserves cross ratios)."

Instead of point O - the point lying on both lines l_1, l_2, we will have the line P_1P_2 that belongs to both pencils.

Finally, we should understand what is dual to a central projection with center C. Recall that it sends the point X on line l_1 to the point of l_2 lying on the line CX.

We now define a "central projection from pencil of lines $\mathfrak{P}_1$ to the pencil of lines $\mathfrak{P}_2$ with center at line l". The image of line $X \in \mathfrak{P}_1$ under this projection will be the line in $\mathfrak{P}_2$ passing through point of intersection of X with l.

Finally we can formulate the dual statement to the end:

Let $f : \mathfrak{P}_1 \to \mathfrak{P}_2$ be a mapping from the pencil $\mathfrak{P}_1$ of lines passing through point P_1 to the pencil $\mathfrak{P}_2$ of lines passing through point P_2. Suppose that f is projective (i.e. preserves cross ratios). Suppose also that $f(P_1P_2) = P_1P_2$. Then there exists a line l in the plane so that the mapping f coincides with a mapping sending line $X \in \mathfrak{P}_1$ to the line in $\mathfrak{P}_2$ passing through point of intersection of X with l.

We hope the reader now has acquired some sense for what it means to dualize statements/definitions of projective geometry.

1.5 Projective mappings from a line to a line, revisited

Now we have enough tools to describe all projective mappings from a line l_1 in a plane to another line l_2 in the same plane.

Recall that we have already proved that through any five points one can draw a unique quadric provided no three points among the five are collinear. Moreover, in this case the quadric is non-degenerate.

Duality gives us the fact that there exists exactly one quadric tangent to given five lines in general position (i.e. no three meet at a point), and, moreover, this quadric is non-degenerate.

Now we will prove the following theorem.

Theorem 3. *Let $f : l_1 \to l_2$ be a projective mapping from line l_1 to line l_2 in the plane. Let O be the point of intersection of l_1 and l_2.*

If $f(O) = O$, then there exists a point C in the plane so that the mapping f coincides with the central projection with center C.

If $f(O) \neq O$, then there exists a non-degenerate conic C tangent to l_1 and l_2 such that the mapping f sends every point X on l_1 to the point on l_2 lying on the line passing through X and tangent to C, which is different from l_1.

Proof. We have dealt with the proof of the first statement in chapter ...

Now suppose $f(O) \neq O$. Take any three points on l_1 which are distinct and do not coincide with O or its preimage under f. Call these points A, B, C.

Now the lines $l_1, l_2, Af(A), Bf(B), Cf(C)$ are five lines in general position. Hence there exists a unique non-degenerate quadric C tangent to all five of them. Now define the map $\tilde{f}$ to be the map that takes a point X on l_1 and outputs the point of l_2 that lies on the tangent line to conic C through point X and which is different from l_1. The map $\tilde{f}$ is projective (see the proof that cross ratio of four points on a quadric is well-defined). Moreover, the maps f and $\tilde{f}$ agree on three points: A,B,C! Hence f must be the same as $\tilde{f}$. $\square$

Chapter 2

Spherical Geometry

The geometry we study in this chapter is the real geometry in the original meaning of the word: measuring the Earth. The surface of the Earth is in some approximation a sphere and the shortest way to go from one point on it to another is not a straight line, but rather a curve on the sphere. These curves are usually called geodesics.

The geodesics on a sphere are parts of great circles, i.e. the circles which are the intersections of the sphere with a plane passing through the center of the sphere. For instance the shortest way to fly from a point on the equator of the Earth to another point on the equator is along the Equator. We won't prove it in this lecture, however.

Instead we will define the spherical geometry in the way Klein would: the space we will be dealing with will be the sphere of radius R centered at the origin of an Euclidean space and the group of isometries of spherical geometry will be the group of rotations around the origin.

We will think of the great circles as of the lines in spherical geometry. Through any two points on the sphere we can draw a unique line in this sense, except for pairs of points, which are antipodes of each other, like North and South pole. For such points there is an infinite family of lines passing through them. Indeed, if the two points on the sphere are not opposite to each other, then they define a unique plane containing them and the origin; the intersection of this plane with the sphere is the great circle we are interested in. If the points are opposite to each other, instead of one plane we get a pencil of planes.

2.1 Area in Spherical Geometry

Now we would like to study the simplest figures on the sphere. Triangles were the simplest figures on a plane, but on a sphere we have a figure which is even simpler - a slice. A slice is the figure formed by two lines on the sphere. Note that on a sphere any two lines intersect at two points which are opposite to each other. The only parameter that a slice has is the angle formed by the two lines defining the slice (recall that angle between any two curves is defined as the angle between the tangent lines to these curves at the point of intersection; alternatively we can define an angle between two great circles on the sphere as the angle between the two planes that contain these great circles).

Let's consider a slice of angle α and compute it's area. It's pretty obvious that the area of such a slice should be proportional to the angle α. Now if we take $\alpha = \pi$, the slice is half-sphere, and hence it should have area $2\pi R^2$. Hence we get that the area of a slice of angle α is $2\alpha R^2$.

Now let's consider a spherical triangle: a triangle cut out by three great circles. Let α, β, γ be its angles. The two slices of angle α together with the two slices of angle β and together with two slices of angle γ cover the entire sphere exactly once, except that they cover the two triangles they all cover three times. Thus the area of the whole sphere + the area of four triangles is the same as the sum of the areas of the six slices: $4\pi R^2 + 4A = 4\alpha R^2 + 4\beta R^2 + 4\gamma R^2$, or $A = (\alpha + \beta + \gamma - \pi)R^2$, where A denotes the area of the triangle.

The result we got is very different from the sort of results we have in Euclidean geometry - in Euclidean geometry the sum of angles of a triangle is always π. In spherical geometry, however, the difference between the sum of angles and π is proportional to the area of triangle! The statement that the sum of angles of planar triangle is π follows in fact from what we proved for the spheres. Imagine a huge sphere of radius R lying on a plane and imagine a triangle drawn on the plane. The spherical triangle obtained from it by central projection with center at the center of the sphere from the plane to the sphere is very close to the planar triangle, when R is big enough. But then the difference of sum of angles of the spherical triangle and π is equal to the area divided by R^2, so as R tends to infinity, the spherical triangle approaches the planar one, the area approaches the area of planar triangle and thus the difference of sum of angles and π approaches zero.

The formula we proved could be used in principle to measure the radius of the Earth without using anything which doesn't lie on its surface. Indeed, if we could draw a big triangle on the surface of our sphere, measure its area accurately and measure its angles accurately, we would be able to find the radius of Earth.

In fact ancient Greeks did know the radius of Earth with great precision. The reader is challenged to invent several practical ways to measure it.

We can generalize the formula we obtained for spherical triangle to a spherical n-gon: all we have to do is cut the n-gon to $n-2$ triangles and then use that both the areas and the sums of the angles are additive (i.e. the area/sum of angles of the polygon is the same as the sum of areas/sums of angles of all the triangles in its triangulation). If we do so, we get that $A = (\text{sum of angles} - (n-2)\pi)R^2$.

The formula we obtained for an n-gon could be generalized to a formula for an arbitrary piecewise smooth curve on the sphere (we can think of such curves as "n-gons with infinite number of sides). The n that appears in this formula prevents us ,however, from doing so. Our first task then would be to rewrite the formula without n appearing explicitly in it. We can write for instance that $A = (\sum_i \alpha_i - \pi + 2\pi)R^2$, where α_i denote the angles of the n-gon. Now we can interpret the angles $\beta_i = \pi - \alpha_i$ as the exterior angles of the n-gon, and thus we get the formula $A = (2\pi - \text{sum of exterior angles})R^2$.

In fact this last formula, once interpreted correctly, can be generalized to arbitrary two-dimensional surfaces! The quantity Area$/\pi R^2$ should be interpreted as the total amount of curvature enclosed by a curve (where curvature is a quantity that can be computed at any point of the surface and it expresses the local geometry of the surface near this point), the sum of exterior angles should be interpreted as the total amount by which a vector rotates, when moved in a parallel way along the curve, and finally the number 2 should be replaced by 2-the number of handles the surface has. This formula, known as Gauss-Bonnet formula, is one of the most basic facts in differential geometry of two-dimensional surfaces.

2.2 Bisectors, Medians and Heights in Spherical Geometry

We would like to prove the analogues of theorems about bisectors, medians and heights being concurrent in spherical geometry. The notions of bisectors, medians and heights are easy to define: we do know what angles and distances are in spherical geometry.

In fact the proof of the claim that angle bisectors are concurrent can be repeated word-for-word from the corresponding planar proof. The crucial fact is that angle bisector is the locus of points equidistant from the two sides of the angle. Once we know this fact, we can take the point of intersection of two bisectors - it is equidistant from all three sides of the triangle. In particular it must lie on the third angle bisector as well.

To prove the corresponding theorems about medians and heights we will use a different tool, which we didn't have in planar geometry.

2.2.1 Duality in Spherical Geometry

Duality in spherical geometry is reminiscent of what we had in projective geometry - it interchanges points and lines, collinearity and concurrency. In fact the duality in spherical geometry is even easier to visualize than that for projective geometry.

We call line on a sphere dual to a point if the plane containing the line is orthogonal to the vector from the center of the sphere to the point on the sphere. Thus for every line there are two opposite points dual to them and for any point there is exactly one line dual to it.

If three points lie on a line, then the three dual lines intersect at a pair of opposite points. This pair of points is the pair of points dual to the line on which the original three points lie. To see this, imagine a point rotating along a great circle. Then the dual plane is rotating around the axis connecting the two dual points to this great circle.

Conversely, if three lines intersect at a point, then the six points dual to them (which consist of three pairs of opposite points) all lie on one line.

To make the treatment of duality a little bit more symmetric (so that a point will be dual to a line and a line to a point, not to a couple of them, we can choose an orientation on a sphere and then define duality between points and oriented lines. Let us ignore this details for now.

To deal with duality in concrete terms, we can use the notion of cross-product of vectors. Recall that the cross-product of two vectors v and w is the vector $v \times w$ which is orthogonal to v and w and whose length is equal to the area of parallelogram spanned by v and w. The direction of this vector can be determined by a right- or left- hand rule, depending on the orientation chosen for the ambient three-dimensional space. In fact if we compute the cross-product of v and w in the other order, we get the same result, except pointing in opposite direction: $w \times v = -v \times w$.

Other properties of cross-product include bilinearity:

$v \times (w_1 + w_2) = v \times w_1 + v \times w_2$ (also $(v_1 + v_2) \times w = v_1 \times w + v_2 \times w$) ,

$v \times (\lambda w) = (\lambda v) \times w = \lambda(v \times w)$

and the Jacobi identity

$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$

This identity expresses the fact that cross-product is not associative, but is not too far from being associative (in the sense that associativity gets replaced by the Jacobi identity). Jacobi identity in fact appears in the theory of Lie algebras - infinitesimal motions or symmetries - and there it plays a fundamental role. In fact the Jacobi identity for the cross product is related to the Lie algebra of the group of all rotations of three-dimensional space.

To verify that this identity is true, one can verify it for u, v, w - vectors of some orthonormal basis and then use the bilinearity of the cross product to deduce it for any vectors. We will leave this verification to the reader.

Now how is this notion of cross-product useful for our purposes? Let A, B denote two points on a sphere, which are not opposite to each other. We will identify these points with the vectors from the origin to them. The vector $A \times B$ is a vector which is orthogonal to both A and B. It is not lying on the sphere, but nevertheless it expresses the direction of axis, on which the points dual to the line through A and B lie (in fact the vector $\frac{A \times B}{|A \times B|}$ is exactly the point dual to the line AB).

Let's apply this idea to proving that three medians in a triangle are concurrent. To prove this statement it is enough to show that the points dual to the medians are collinear. Now the midpoint of the segment AB can be expressed as $\frac{A+B}{|A+B|}$ - since vectors A and B have equal length, the direction $A + B$ in plane through the origin, A and B which points towards the midpoint between A and B.

Thus the point dual to the median from vertex C, i.e. to the line connecting C with the midpoint of AB is proportional to $(A+B) \times C$. Similarly the points dual to other medians are proportional to $(B+C) \times A$ and $(C+A) \times B$ respectively. Now notice that antisymmetry of the cross-product implies that the sum $(A+B) \times C + (B+C) \times A + (C+A) \times B$ is equal to zero. This means that the origin and the points $(A+B) \times C$, $(B+C) \times A$ and $(C+A) \times B$ are coplanar, which implies the collinearity (on the sphere) of the points dual to the medians.

The story is similar for the heights: the height from point C must be orthogonal to the line joining A and B, hence it must contain the point dual to line AB, i.e. the point $\frac{A\times B}{|A\times B|}$. Thus the point dual to this height must be orthogonal both to the direction $A \times B$ and to C. So the point dual to height from vertex C is proportional to $(A \times B) \times C$. Similarly the points dual to the other heights are proportional to $(B \times C) \times A$ and $(C \times A) \times B$ respectively. Now notice that Jacobi identity for the cross-product implies that the sum $(A\times B)\times C+(B\times C)\times A+(C\times A)\times B$ is equal to zero. Like in the previous proof, this means that the origin and the points $(A\times B)\times C$, $(B \times C) \times A$ and $(C \times A) \times B$ are coplanar, which implies the collinearity (on the sphere) of the points dual to the heights.

2.3 Bonus: a Funny Proof of Pascal's Theorem

(to be moved to the section on Pascal's theorem)

Let's go back to Pascal's theorem and try to prove it without mentioning the notion of cross-ratio. Our strategy this time will be as follows:

Let $L_1, \ldots, L_6$ denote the equations of the six sides of the hexagon inscribed into a conic (i.e. L_i is a linear function that vanishes exactly on the i-th side of the hexagon). We will prove that there exists a number λ such that $L_1L_3L_5 = \lambda L_2L_4L_6$ for every point on the conic. This will imly that the quadratic polynomial Q must divide the cubic polynomial $L_1L_3L_5-\lambda L_2L_4L_6$. The quotient, L, will be a linear function that vanishes at every point not on the quadric where $L_1L_3L_5 - \lambda L_2L_4L_6$ vanishes. In particular it will vanish at points of intersection of L_1 with L_4, of L_2 with L_5 and of L_3 with L_6. Thus these points of intersection all lie on the line $L = 0$, which proves Pascal's theorem.

The only part that requires a further proof is the existence of number λ so that $L_1L_3L_5 = \lambda L_2L_4L_6$ along the conic.

First we present a very simple proof that relies, however, on knowledge of complex analysis. Consider all the equations we have as equations on complex numbers that define figures in complex plane. Also make also all the equations homogeneous and consider them in the complex projective plane. Then the quadric becomes isomorphic to the Riemann sphere - this follows from the fact that there exists a rational parametrization of the quadric and that the quadric is smooth. The function $\frac{L_1L_3L_5}{L_2L_4L_6}$ is in fact a function on the complex projective plane, since the numerator and the denominator are homogeneous of the same degree (3). Finally, when restricted to the quadric, this function doesn't have any poles - the zeros of the denominator get canceled with the zeros of the numerator (they both are at the vertices of the hexagon). Thus Liouville's theorem implies that the function must be constant: $\frac{L_1L_3L_5}{L_2L_4L_6} = \lambda$.

While this proof is quite transparent, we would like to give an alternative proof that uses considerations with real numbers only. This way we will have a real (in the sense of real numbers) proof of Pascal's theorem, which the readers unfamiliar with complex analysis will be able to understand.

Since Pascal's theorem is invariant under projective transformations, it is enough to prove it for the case when the quadric is a circle. For the circle we will choose the linear function vanishing on the side L_i to be the function of distance to the side L_i (we will think of this distance as positive on one side of the line L_i and negative on the other side, so that it becomes a genuine linear function).

We will prove the following more general statement:

Lemma 4. *Let $L_1, \ldots, L_{2n}$ denote the sides of a $2n$-gon inscribed in a circle. Let P be a point of the circle and let h_i denote the distance from O to L_i. Then $h_1h_3 \ldots h_{2n-1} = \pm h_2h_4 \ldots h_{2m}$ (the sign $\pm$ depends on the choices where the distances are positive and where they are negative).*

Proof. If A,B and C are three points on a circle of radius R, then the length of side AB is equal to $2R \sin \angle ACB$, and thus the length of the height from A to the side BC is equal to $2R \sin \angle ACB \sin \angle ABC$.

If we denote the vertices of the $2n$-gon inscribed in the circle $A_1, \ldots, A_{2n}$ (so that h_i is the length of the height from point P on the side A_iA_{i+1}), then the product $h_1h_3 \ldots h_{2n-1}$ can be expressed as

$$(2R \sin \angle PA_1A_2 \sin \angle PA_2A_1)(2R \sin \angle PA_3A_4 \sin \angle PA_4A_3) \ldots (2R \sin \angle PA_{2n-1}A_{2n} \sin \angle PA_{2n}A_{2n-1})$$

Similarly the product $h_2h_4 \ldots h_{2n}$ can be expressed as

$$(2R \sin \angle PA_2A_3 \sin \angle PA_3A_2)(2R \sin \angle PA_4A_5 \sin \angle PA_5A_4) \ldots (2R \sin \angle PA_{2n}A_1 \sin \angle PA_1A_{2n})$$

But now we notice that the angles $\angle PA_{i-1}A_i$ and $\angle PA_{i+1}A_i$ are always either equal or complimentary (i.e. sum up to π), so that their sines are equal. □