University of Toronto
FACULTY OF ARTS AND SCIENCE
Final Examinations, ..., 2004

MAT325H Classical Plane Geometries and their Transformations
Instructor: Prof. A. Khovanskii
Duration – 3 hours

NO AIDS ALLOWED
Total marks = 100. All questions have equal value

Problem 1.

Prove the Menelaus's theorem:

Take three lines l_1, l_2, l_3 and consider three points L, M, N at them: $L \in l_1$, $M \in l_2$, $N \in l_3$. Assume that A, B, C are points of intersections of these lines: $A = l_1 \cap l_2$, $B = l_2 \cap l_3$, and $C = l_3 \cap l_1$.

Points L, M, N belong to one line, if and only if

$$\frac{AL}{CL} \cdot \frac{BM}{AM} \cdot \frac{CN}{BN} = 1.$$

Problem 2.

Consider a regular triangle ABC. Find all points O for which the sum $4O_{AB} + O_{BC} + O_{CD}$ is the smallest possible. Here O_{AB}, O_{BC} and O_{CD} are the distances from point O to the sides AB, BC and CD respectively.

Problem 3.

Consider a square $ABCD$ inscribed in a circle. Let P be an arbitrary point on the circle. Explain why the cross-ratio of the lines AP, BP, CP, and DP is independent of the choice of point P. Find this cross-ratio.

Problem 4.

Consider two circles S_1, S_2 with centers O_1, O_2 and radiuses R_1, R_2. Make inversion with respect to the circle S_1 and then make inversion with respect to the circle S_2. Describe all lines and circles which after two inversions will become straight lines. (Hint: to start with describe all lines and circles which become straight lines after one inversion with respect to the second circle S_2.)

Problem 5.

Consider a sphere S of radius R. Cover it by equal triangles assuming that each angle of each triangle equals to $2\pi/5$ and assuming that two different triangles or have a common vertex or have a common side or have no points of intersections. How many triangles are there in such covering? (Hint: use areas)