THE FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2006

MAT402H1S
Classical Geometries

Examiner: Professor A. Khovanskii
Duration: 3 hours

NO AIDS ALLOWED. **Total Marks: 100**

1. *[20 marks]*
Take a convex polyhedron in $\mathbf{R}^3$. Denote by f_0, f_1 and f_2 the number of its vertices, edges and faces, respectively. Prove:
1) $3f_0 \leq 2f_1$. Hint: at least 3 edges meet at each vertex of the polyhedron.
2) $2f_1/f_2 < 6$ — the average number of edges on faces of the polyhedron is strictly less that 6. Hint: use 1) and Euler formula $f_0 - f_1 + f_2 = 2$.

2. *[20 marks]*
Take an angle between 2 rays l_1 and l_2 with vertex O and a point A inside the angle. Consider all triangles with vertex O such that two sides of each of them belong to l_1 and l_2 and the third side l passes through A. Find the line l for which the area of the triangle is minimal. Hint: consider the parallelogram with two sides in l_1 and l_2 and with center A and look at how the line l cuts this parallelogram.

3. *[20 marks]*
Consider a square $ABCD$ inscribed in a circle. Let P be an arbitrary point on the circle. Explain why the cross-ratio of the lines AP, BP, CP, and DP is independent of the choice of point P. Find this cross-ratio.

4. *[20 marks]*
Consider two circles S_1, S_2 with centers O_1, O_2 and radiuses R_1, R_2. Make inversion with respect to the circle S_1 and then make inversion with respect to the circle S_2. Describe all lines and circles which become straight lines after these two inversions.

5. *[20 marks]*
Consider a sphere S of radius R. Is it possible to locate 50 equal triangles with angles equal to $\pi/2, \pi/3, \pi/4$ on it in such a way that any two triangles do not overlap each other?