

THE FACULTY OF ARTS AND SCIENCE  
University of Toronto

FINAL EXAMINATIONS, APRIL 21, 2009

MAT402H CLASSICAL PLANE GEOMETRIES  
AND THEIR TRANSFORMATIONS

Examiner: Professor A. Khovanskii

Duration: 3 hours

**NO AIDS ALLOWED.**

**Total Marks: 100**

1. [20 marks] Consider angle  $\alpha = 45^\circ$  between two rays  $l_1$  and  $l_2$  intersecting at point  $O$ . Take any point  $A$  inside the angle. Find points  $B \in l_1$  and  $C \in l_2$  such that polygonal path  $ABCA$  has the smallest length. Find this smallest length assuming that the distance from  $A$  to  $O$  is  $a$ .
2. [20 marks] Consider a triangle  $ABC$ . Let  $D$  be the point on the side  $AB$  such that  $AD : DB = 2$  and let  $E$  be the point on the segment  $CD$  such that  $DE : EC = 2$ . In what proportion the line  $AE$  divides the side  $CB$ ? In what proportion the line  $BE$  divides the side  $CA$ ?  
Hint: Put appropriate masses at the points  $A$ ,  $B$  and  $C$ .
3. [20 marks] Consider triangle  $ABC$  such that  $AB = 3$ ,  $BC = 4$ ,  $CA = 5$ . Find the point  $O$  such that after an inversion centered at  $O$  the line passing through  $A, C$  becomes a line, and lines passing through  $A, B$  and through  $B, C$  become equal circles.
4. [20 marks] Consider a square  $ABCD$  inscribed in a circle. Take any line  $l$  tangent to the circle, and consider points  $A' = l \cap l_A$ ,  $B' = l \cap l_B$ ,  $C' = l \cap l_C$ ,  $D' = l \cap l_D$  where  $l_A, l_B, l_C, l_D$  are lines tangent to the circle at  $A, B, C, D$ . Explain why the cross-ratio  $(A', B', C', D')$  is independent of the choice of the tangent line  $l$ . Compute this cross-ratio.
5. [20 marks] Take a sphere of radius  $R$  and a (spherical) triangle on this sphere with angles  $\alpha, \beta, \gamma$ . Write a formula for the area of the triangle (in terms of the angles  $\alpha, \beta, \gamma$ ) and prove this formula.