UNIVERSITY of TORONTO

MAT402H CLASSICAL PLANE GEOMETRIES

AND THEIR TRANSFORMATIONS

March 13, 2008 (from 6:10 to 9:00 p.m.)

Term test

No aids allowed.

Problem 1 (20 points).

Prove that in any non-isosceles triangle ABC the three points of intersection of the bisectors of its external angles with the opposite sides belong to one line.

Hint: If P is the point of intersection of the bisector of the external angle A with the extension of the side BC, then PC:PB= AC:AB. Prove it (using similar arguments to our proof of a similar statement for the interior bisector).

Problem 2 (20 points for p. 1 or 10 points for p. 2).

1) Prove Radon's Theorem in $\mathbb{R}^d$: Assume that we have a collection of points $A \subset \mathbb{R}^d$ such that their number is $d+2$. Then there is a subset $B \subset A$ such that the convex hull of (B) and convex hull of $(A \setminus B)$ have a non empty intersection.

2) Prove Radon's theorem for $d = 2$.

Problem 3 (20 points).

Take a couple of parallel segments AD and BC of lengths a and b respectively, where $a > b$. Consider a trapezoid ABCD. Let P be the point of intersection of the lines containing the sides AB and DC. Let Q be the point of intersection of the diagonals of the trapezoid. Prove that the line PQ intersects the sides BC and AD at their midpoints.

Hint: put masses at the points A, P and D in such a way that the center of masses would be located at the point Q.

Problem 4 (20 points).

Consider a triangle ABC. Assume that angles at the vertexes A, B are smaller than 45 degrees. Take any point P inside the triangle. Find points $C' \in CB$, $B' \in BA$ and $A' \in AC$ for which the sum $PC' + C'B' + B'A' + A'P'$ will be the smallest.

Problem 5 (20 points).

Take a regular triangle inscribed into a circle. Describe the image of the triangle (including its interior) under inversion with respect to the circle.