UNIVERSITY of TORONTO

MAT402H CLASSICAL PLANE GEOMETRIES

AND THEIR TRANSFORMATIONS

March 12, 2009 (from 6:10 to 9:00 p.m.)

Term test

No aids allowed.

Problem 1 (20 points).

Consider a regular triangle ABC. Find all points O in the triangle for which the sum $2O_{AB} + 2O_{BC} + O_{CA}$ is the biggest possible. Here O_{AB}, O_{BC} and O_{CA} are distances from point O to the sides AB, BC and CA respectively.

Problem 2 (20 points).

Consider a tetrahedron in $\mathbb{R}^3$. Mark a point at the middle of each side of the tetrahedron. Join by a segment the marked points belonging to the opposite sides. Prove that the three segments in $\mathbb{R}^3$ we constructed pass through one point. In what proportion the intersection point divides each segment?

Hint: put appropriate masses at the vertices of the tetrahedron and use uniqueness of the center of masses.

Problem 3 (20 points). Consider plane $\mathbb{R}^2$ as complex plane. Consider a transformation $z \to \frac{z+1}{z-1}$. Which circles under this transformation will become lines?

Problem 4 (20 points).

Take two circles S_1 and S_2 intersecting at points A and B. Consider all circles S orthogonal to S_1 and to S_2. Find the locus of centers of all such circles S.

Hint: How the points A and B are located with respect to a circle S?

Problem 5 (20 points).

Prove the Menelaus's theorem:

Take three lines l_1, l_2, l_3 and consider three points L, M, N on them: $L \in l_1$, $M \in l_2$, $N \in l_3$. Assume that A, B, C are points of intersections of these lines: $A = l_1 \cap l_2$, $B = l_2 \cap l_3$, and $C = l_3 \cap l_1$.

Points L, M, N belong to one line, if and only if

$$\frac{AL}{CL} \cdot \frac{BM}{AM} \cdot \frac{CN}{BN} = 1.$$